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Invariant Subspaces of L^∞ of Certain Homogeneous Spaces

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It is proved that $L^\infty(G/H)$ does not contain any proper G-invariant closed subspaces of finite codimension, where the hypotheses are that G is a locally compact group, H a closed subgroup such that G/H is compact, and such that $\Delta_G|_H \neq \Delta_H$, where Δ denotes the modular function of the group involved.

THEOREM 1. *Let G be a locally compact group and let H be a closed subgroup such that G/H is compact. Suppose there is a $p_0 \in H$ for which $\Delta_H(p_0) \neq \Delta_G(p_0)$, where Δ_H and Δ_G are the modular functions of H and G, respectively. Suppose W is a closed subspace of (complex) $L^\infty(G/H)$ that is invariant under the natural action of G on $L^\infty(G/H)$ and such that W has finite codimension in $L^\infty(G/H)$. Then $W = L^\infty(G/H)$.*

This theorem overlaps a result [5, Theorem 1] of Rubel and Shields which considers invariant subspaces of $L^\infty(T)$, where $T = \{z: |z| = 1\}$ is the unit circle, under the action of the Möbius group M of all maps

$$\mu : z \mapsto e^{i\lambda} \frac{z - z_0}{1 - \bar{z}_0 z}, \lambda \in \mathbf{R}, |z_0| < 1.$$

Here, $G = M$ is isomorphic to $PSL(2, \mathbf{R})$, and is unimodular because it is simple. Let H be the subgroup of all elements of G that leave the point $z = 1$ fixed. Because G is transitive, we see that G/H may be taken as T, and G acts on G/H the way M acts on T. Now H is isomorphic to the group of matrices of the form,

$$m = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}; \qquad a > 0, \quad b \text{ real},$$

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so that [1, Chapter 7, p. 84] $\Delta_H(m) = a^2$. Hence the considerations of [5, Theorem 1] satisfy the hypotheses of our Theorem 1. However [5] considers a large class of subspaces of $L^\infty(T)$, while we restrict our attention to $L^\infty(T)$ itself.

Our theorem is closely related to a theorem of Weil [6, p. 45] on the existence of invariant measures; where it is proved that the condition that Δ_G coincide with Δ_H on H is necessary and sufficient that G/H possess an invariant countably additive Borel measure. Our theorem may be considered as a generalization to the case of finitely additive measures, when G/H is compact. However when G/H is not compact, there may exist finitely additive invariant means even when countably additive ones do not exist. For example, when G is solvable, it is known [2, Theorem 1.2.1, p. 5; Theorem 1.2.6, p. 8; Theorem 2.2.1, p. 26] to be amenable, so that if H is a closed subgroup, then G/H admits an invariant mean.

To say that a locally compact group is amenable is to say that there exists an invariant mean on $L^\infty(G)$. For example, let

$$G = \left\{ \begin{pmatrix} x & Y \\ 0 & 1 \end{pmatrix} : x > 0,\ y \text{ real} \right\},$$

$$H = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x > 0 \right\}.$$

Then G is solvable, $\Delta_G|_H \neq \Delta_H$ and G/H admits an invariant mean. To show that we need $\Delta_G|_H \neq \Delta_H$, take $G = \mathbf{R}$ and $H = \mathbf{Z}$ so that $G/H = T$, where G acts via rotation. Any character of T gives rise to an invariant subspace of codimension 1. In [5, Theorem 2], it was shown that there are actually invariant subspaces of codimension 1 in $L^\infty(T)$ that contain all the continuous functions. So the conditions on the modular functions and on the compactness of G/H are each needed.

After we proved Theorem 1, H. Furstenberg has found a different proof.

Proof of the Theorem. We suppose here only that G is a locally compact group and that H is a closed subgroup of G. By $C_c(G)$ we denote the space of all continuous complex-valued functions on G with compact support. The notions $C_c(G/H)$ and $L_c^\infty(G/H)$ are similarly used to denote compact support of the functions involved. If μ is a linear functional on $C_c(G/H)$ then $r(p)\mu$, for $p \in H$, means the right translate of μ by p, i.e.,

$$(r(p)\mu)(f) = \mu(r(p^{-1})f).$$

For $x \in G$ and μ as before, we mean by $x \cdot \mu$ the functional

$$(x \cdot \mu)(f) = \mu(f_x),$$

where

$$f_x(t) = f(x^{-1}t).$$

LEMMA 1. *Let $\mu_1, \mu_2, \ldots, \mu_N$ be nonnegative linear functionals on $C_c(G)$ such that*

(i) $r(p)\,\mu_i = \Delta_H(p)\,\mu_i$, $i = 1, 2, \ldots, N$,

(ii) $x \cdot \mu_i \leqslant M \sum_{j=1}^{n} \mu_j$, *for all* $x \in G$,

where M is a constant. If $\Delta_H(p_0) \neq \Delta_G(p_0)$ for some $p_0 \in H$, then $\mu_i = 0$ for all $i = 1, 2, \ldots, N$.

Proof. Choose such a p_0 and let ω be any compact neighborhood in G of p_0. Let G_0 be the subgroup of G generated by ω, and let $H_0 = G_0 \cap H$. Then G_0 and H_0 are open (and hence closed) subgroups of G and H, respectively. So $\Delta_G = \Delta_{G_0}$ on G_0 and $\Delta_H = \Delta_{H_0}$ on H_0. Note that G_0 is σ-compact. The restrictions of the μ_i to $C_c(G_0)$ satisfy the conditions (i) and (ii). Since we may take ω to contain any given compact set, the result is proved if we can prove it for G_0. In short, we may assume without loss of generality that G is σ-compact, and we do so. Let $\lambda = \sum \mu_i$. Then λ is finite, and for $M' = NM$ we have

$$x \cdot \lambda \leqslant M'\lambda.$$

So by a theorem of Mackey [3, Theorem 1.1, p. 106], λ is absolutely continuous with respect to Haar measure on G, since the null space of λ is translation invariant. Hence by the Radon–Nikodym theorem [4, p. 238] there exists a locally summable function ζ on G such that

$$\lambda(f) = \int \zeta(x) f(x)\, dx,$$

for $f \in C_c(G)$. The relation $x \cdot \lambda \leqslant M'\lambda$ implies that for each x, $\zeta(x^{-1}y) \leqslant M'\zeta(y)$ for almost all y. So by Tonelli's theorem [4, p. 270], there exists $y_0 \in G$ such that $\zeta(x^{-1}y_0) \leqslant M'\zeta(y_0)$ for almost all x. Thus $\|\zeta\|_\infty < \infty$. Relation (i) now implies that,

$$\|\zeta\|_\infty = (\Delta_H(p_0)/\Delta_G(p_0))\,\|\zeta\|_\infty, \qquad \text{so that} \quad \|\zeta\|_\infty = 0.$$

Hence $\lambda = 0$ and so each $\mu_i = 0$.

LEMMA 2. *Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be a finite number of nonnegative linear functionals on $C_c(G/H)$ such that*

$$x \cdot \lambda_i \leqslant M \sum_{j=1}^{N} \lambda_j,$$

for all $x \in G$. *If, for some* $p_0 \in H$,

$$\Delta_H(p_0) \neq \Delta_G(p_0),$$

then $\lambda_j = 0$ *for all* j.

Proof. Let μ_i be the linear functionals defined on $C_c(G)$ by

$$\mu_i(f) = \lambda_i(\bar{f}),$$

where

$$\bar{f}(xH) = \int_H f(xh)\, d_l h.$$

Now [6, p. 43] $f \mapsto \bar{f}$ is a linear and onto map of $C_c(G)$ onto $C_c(G/H)$, so $\lambda_i = 0$ if $\mu_i = 0$. But the μ_i satisfy the hypotheses of Lemma 1, and we are done.

Proof of the Theorem (Completed). Let $W^{\perp}$ be the space of continuous linear functionals on $L^{\infty}(G/H)$ that annihilate W. Let $\nu_1, \nu_2, \ldots, \nu_N$ be a basis for $W^{\perp}$. The ν_i are continuous linear functionals on $L^{\infty}(G/H)$. There exist $f_i \in L^{\infty}(G/H)$, $i = 1, 2, \ldots, N$ such that $\nu_i(f_j) = \delta_{ij}$. It follows that

$$x \cdot \nu_i = \sum_{j=1}^{N} C_{ji}(x)\nu_j ,$$

where

$$C_{ji}(x) = \nu_i((f_j)_{x^{-1}})$$

and the C_{ji} are clearly bounded, say

$$| C_{ji}(x)| \leqslant M, \qquad x \in G.$$

Let $| \nu_j |$ denote the variation of ν considered as a linear functional on $L^{\infty}(G/H)$ and let λ_j be the restriction of $| \nu_j |$ to $C_c(G/H)$.

In symbols, $\lambda_j = | \nu_j | |_{C_c(G/H)}$. Now

$$x \cdot \lambda_i = | x \cdot \nu_i | |_{C_c(G/H)} = \left| \sum C_{ji}(x)\nu_i \right| |_{C_c(G/H)}$$

$$\leqslant M \sum_j | \nu_j | |_{C_c(G/H)} = M \sum_j \lambda_j .$$

By Lemma 2, $\lambda_j = 0$ for all j. Therefore, $W \supseteq L_c^{\infty}(G/H) = L^{\infty}(G/H)$ since G/H is supposed compact, and the theorem is proved.

We mention that if we do not assume that G/H is compact, we may still conclude under the remaining hypotheses of Theorem 1 that $W \supseteq L_c^{\infty}(G/H)$. We remark that Theorem 1 implies that if G is a locally compact group for which there exists a closed subgroup H satisfying the hypotheses of Theorem 1, then G is not amenable. Consequently we have a new proof that a semisimple Lie group is not amenable.

We conclude with a problem. Suppose, under the hypotheses of Theorem 1 that E and F are closed invariant subspaces of $L^\infty(G/H)$, that $E \subseteq F$, $C(G/H) \subseteq F$, and that F is a module over $C(G/H)$. If $\dim F/E < \infty$, does it follow that $E = F$?

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